REPRESENTATIONS OF REDUCTIVE GROUPS OVER QUOTIENTS OF LOCAL RINGS

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ABSTRACT. In some recent work, Lusztig outlined a generalisation of the construction of Deligne and Lusztig to reductive groups over finite rings coming from the ring of integers in a local field, modulo some power of the maximal ideal. Lusztig conjectures that all irreducible representations of these groups are contained in the cohomology of a certain family of varieties. We show that, contrary to what was expected, there exist representations that cannot be realised by the varieties given by Lusztig. Moreover, we show how the remaining representations in the case under consideration can be realised in the cohomology of a different kind of variety. This may suggest a way to reformulate Lusztig's conjecture.

1. INTRODUCTION

Let G be a connected reductive group, defined over a finite field $\mathbb{F}_q$ of characteristic p. The celebrated work of Deligne and Lusztig ([2]) uses methods of l-adic étale cohomology to show that many complex irreducible representations of the finite group $G(\mathbb{F}_q)$ are parametrised by characters of maximal tori in $G(\mathbb{F}_q)$, and moreover, that every irreducible representation appears in the cohomology of a certain kind of variety.

Now let K be a local field (of any characteristic) with finite residue field $\mathbb{F}_q$, and ring of integers $\mathcal{O}_K$. Fix an algebraic closure of K, and let $\mathcal{O}$ be the ring of integers of the maximal unramified extension K^{ur} of K with residue field $\mathbb{F}$, an algebraic closure of $\mathbb{F}_q$. Denote by ε a fixed prime element in $\mathcal{O}_K$. It is also a prime element in $\mathcal{O}$.

Assume that X is an affine variety over $\mathbb{F}$, and let $r \geq 1$ be an integer. We set

$$X_r = X(\mathcal{O}/(\varepsilon^r)).$$

Thus, if X is the common zeroes of the polynomials $f_i(x_1, \dots, x_n)$, $i = 1, \dots, m$, then X_r is the set of all $(a_1, \dots, a_n) \in (\mathcal{O}/(\varepsilon^r))^n$ such that $f_i(a_1, \dots, a_n) = 0$ for $i = 1, \dots, m$. This makes sense, since $\mathcal{O}/(\varepsilon^r)$ is an $\mathbb{F}$-algebra. Then $X \mapsto X_r$ is a functor from the category of affine varieties over $\mathbb{F}$ into itself.

For any $r \geq r' \geq 0$ the reduction map $\mathcal{O}/(\varepsilon^r) \to \mathcal{O}/(\varepsilon^{r'})$ induces a morphism $\rho_{r,r'} : X_r \to X_{r'}$. We will denote the map $\rho_{r,1}$ by ρ_r. Consider the reductive algebraic group G over $\mathbb{F}$. Then G_r is naturally an algebraic group over $\mathbb{F}$, and $\rho_{r,r'} : G_r \to G_{r'}$ is a surjective homomorphism of algebraic groups with kernel $G_r^{r'}$. Thus we have an exact sequence

$$1 \longrightarrow G_r^{r'} \longrightarrow G_r \xrightarrow{\rho_{r,r'}} G_{r'} \longrightarrow 1.$$

The injection $\mathcal{O}/(\varepsilon^{r'}) \to \mathcal{O}/(\varepsilon^r)$ induces a function $i_{r',r} : G_{r'} \to G_r$ such that $\rho_{r,r'} \circ i_{r',r}$ is the identity map on $G_{r'}$. In the case where $r' = 1$ and K is a local

field of positive characteristic there is an inclusion of $\mathbb{F}$-algebras $\mathcal{O}/(\varepsilon) \to \mathcal{O}/(\varepsilon^r)$, and $i_{1,r}$ is an injective homomorphism, so that the above exact sequence splits. When K is of characteristic zero, the exact sequence is just a group extension. We will denote $i_{1,r}$ by i_r.

Let $F : G \to G$ be the Frobenius morphism corresponding to the $\mathbb{F}_q$-rational structure of G. The map F induces a homomorphism $F : G_r \to G_r$ which is the Frobenius map for an $\mathbb{F}_q$-rational structure on G_r. Let T be a maximal torus in G contained in an F-stable Borel subgroup B, with unipotent radical U. Fix $r \geq 1$ as above, and let $x \in G_r$. Consider the affine variety

$$X_x = \{g \in G_r \mid g^{-1}F(g) \in xU_r\}.$$

The group G_r^F of fixed points under the Frobenius map can be identified with the $\mathbb{F}_q$-points of G_r, and is thus a finite group. The group G_r^F acts on X_x by left multiplication, and thus by functoriality, the l-adic cohomology with compact support $H_c^i(X_x, \overline{\mathbb{Q}}_l)$ for $l \neq p$, has the structure of a complex representation of G_r^F. Note that $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$ (non-canonically).

The above construction is an extension of the construction of Deligne and Lusztig (which is the case $r = 1$), and was first mentioned by Lusztig in [5], and then recently developed further in [6]. This construction is the first natural step to an extension of the Deligne-Lusztig construction to reductive groups over local fields, a problem which has important arithmetic implications.

In the paper [6], Lusztig proves an orthogonality formula for certain virtual representations of the group G_r^F, in the case where K is a local field of positive characteristic. This result is an extension of an important result of Deligne and Lusztig in the case $r = 1$, and was anticipated in [5].

It was shown in [2] that for the case $r = 1$ every irreducible representation of G^F appears in the cohomology of some variety X_w, where w is an element of the Weyl group of G (the varieties are independent of the lift of w to an element of G). As pointed out in [6], this is no longer the case for $r \geq 2$. In the end of [6], Lusztig states the following

Conjecture. *Any irreducible representation of G_r^F appears in the virtual representation $\sum_{i\geq 0}(-1)^i H_c^i(X_x, \overline{\mathbb{Q}}_l)$ for some $x \in G_r$.*

In the following we will show that the conjecture does not hold for $G = \mathrm{SL}_2$, with K a local field of positive characteristic with q odd, and $r = 2$, i.e., for the group $G_2^F = \mathrm{SL}_2(\mathbb{F}_q[[\varepsilon]]/(\varepsilon^2))$. In the end of this paper we will show that the missing irreducible representations of this group are realised in the cohomology of a certain variety, not of the form X_x for any $x \in G_2$. These representations are parametrised by certain characters of two different subgroups of G_2^F.

So far nothing seems to be known about Lusztig's conjecture in the case where K is a local field of mixed characteristic. This is the focus of work in progress ([8]).

2. The Representations of $\mathrm{SL}_2(\mathbb{F}_q[[\varepsilon]]/(\varepsilon^2))$

Recall the notation of the previous section. We specialise the discussion to the case $G = \mathrm{SL}_2$, K of positive characteristic with q odd, and $r = 2$. The irreducible representations of G_2^F can be classified using the fact that G_2^F is a semidirect product of G^F, and a group N isomorphic to $(\mathbb{F}_q^+)^3$ (three copies of the additive group of the finite field). The following table of representations of G_2^F was given in [6]. The

first column indicates the dimension, and the second column indicates the number of representations of that dimension.

dim	#
1	1
q	1
$q+1$	$(q-3)/2$
$(q+1)/2$	2
$q-1$	$(q-1)/2$
$(q-1)/2$	2
q^2+q	$(q-1)^2/2$
q^2-q	$(q^2-1)/2$
$(q^2-1)/2$	$2q$

A calculation using the fact that the sum of the squares of the dimensions of the irreducible representations of a finite group equals the order of the group, shows that this table is incorrect. We will show in the next section that the correct value for the last entry in the second column is $4q$. We therefore have to consider $2q$ representations in addition to those considered by Lusztig.

Consider G_2^F as a semidirect product of G^F and N, where N is the normal abelian subgroup of G_2^F consisting of matrices of the form

$$\left\{g=\begin{pmatrix}1+a\varepsilon & b\varepsilon\\ c\varepsilon & 1+d\varepsilon\end{pmatrix}\mid \det g=1\right\}=\left\{\begin{pmatrix}1+a\varepsilon & b\varepsilon\\ c\varepsilon & 1-a\varepsilon\end{pmatrix}\right\}.$$

The method of describing the representations of a semidirect product by an abelian group is given in [7]. We give here a summary of this method for the case under consideration.

Let $X=\mathrm{Hom}(N,\mathbb{C}^\times)$. The group G_2^F acts on X by

$$(s\chi)(n)=\chi(s^{-1}ns)\quad\text{for } s\in G_2^F,\chi\in X,n\in N.$$

Let $(\chi_i)_{i\in X/G^F}$ be a system of representatives for the orbits of G^F in X. For each $i\in X/G^F$, let $(G^F)_i=\mathrm{Stab}_{G^F}(\chi_i)$ and let $(G_2^F)_i=(G^F)_i\cdot N$ be the corresponding subgroup of G_2^F. The character χ_i can be extended to $(G_2^F)_i$ by setting

$$\chi_i(gn)=\chi_i(n)\quad\text{for } g\in(G^F)_i,n\in N.$$

Now let ρ be an irreducible representation of $(G^F)_i$. By composing ρ with the canonical projection $(G_2^F)_i\to(G^F)_i$ we obtain an irreducible representation $\tilde\rho$ of $(G_2^F)_i$. Finally, by taking the tensor product of $\tilde\rho$ and χ_i we obtain an irreducible representation $\tilde\rho\otimes\chi_i$ of $(G_2^F)_i$. Let $\theta_{i,\rho}$ be the corresponding induced representation of G_2^F. Then we have the following result (cf. [7], Proposition 25, 8.2.)

Proposition 2.1.

(a) *$\theta_{i,\rho}$ is irreducible.*
(b) *If $\theta_{i,\rho}$ and $\theta_{i',\rho'}$ are isomorphic, then $i=i'$ and $\rho\simeq\rho'$.*
(c) *Every irreducible representation of G_2^F is isomorphic to one of the $\theta_{i,\rho}$.*

We now show how to classify all the irreducible representations of G_2^F of degree $(q^2-1)/2$ using the above method. Let $\psi:\mathbb{F}_q^+\to\mathbb{C}^\times$ be a nontrivial additive

character, and consider the character

$$\chi_\psi : N \longrightarrow \mathbb{C}^\times, \quad \chi_\psi \begin{pmatrix} 1+a\varepsilon & b\varepsilon \\ c\varepsilon & 1-a\varepsilon \end{pmatrix} = \psi(c).$$

Computing the stabiliser of χ_ψ in G^F, we get

$$\begin{aligned} \mathrm{Stab}_{G^F}(\chi_\psi) &= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in G^F \mid \chi_\psi \left[\begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1} \begin{pmatrix} 1+a\varepsilon & b\varepsilon \\ c\varepsilon & 1-a\varepsilon \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right] \right. \\ &= \left. \chi_\psi \begin{pmatrix} 1+a\varepsilon & b\varepsilon \\ c\varepsilon & 1-a\varepsilon \end{pmatrix}, \forall a,b,c \in \mathbb{F}_q^+ \right\} \\ &= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in G^F \mid \psi(x^2c - z^2b + 2xza) = \psi(c), \forall a,b,c \in \mathbb{F}_q^+ \right\} \\ &= \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \in G^F \mid x^2 = 1 \right\} = \left\{ \begin{pmatrix} \pm 1 & y \\ 0 & \pm 1 \end{pmatrix} \mid y \in \mathbb{F}_q \right\}. \end{aligned}$$

Since $\mathrm{Stab}_{G^F}(\chi_\psi)$ is an abelian group of order $2q$, we obtain $2q$ irreducible representations of G_2^F of dimension $(q^2-1)/2$ by inducing certain 1-dimensional representations of $\mathrm{Stab}_{G^F}(\chi_\psi) \cdot N$, according to the method described above. Now the problem is to determine the orbits of all characters of the form χ_ψ under the action of G^F.

Let $\zeta = e^{2\pi i/p}$ and let $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ be the absolute trace. It is well known that every character $\mathbb{F}_q^+ \to \mathbb{C}^\times$ is of the form

$$\psi_a(x) = \zeta^{\mathrm{Tr}(ax)},$$

for some $a \in \mathbb{F}_q$, and that $\psi_a \neq \psi_b$ for $a \neq b$.

We now consider the action of G^F on the characters ψ_a, where $a \in \mathbb{F}_q^\times$. For $k, m \in \mathbb{F}_q^\times$ we see that if χ_{ψ_k} lies in the same orbit as χ_{ψ_m}, then there exists an element $g = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in G^F$such that

$$(g\chi_{\psi_k}) \begin{pmatrix} 1+a\varepsilon & b\varepsilon \\ c\varepsilon & 1-a\varepsilon \end{pmatrix} = \chi_{\psi_m} \begin{pmatrix} 1+a\varepsilon & b\varepsilon \\ c\varepsilon & 1-a\varepsilon \end{pmatrix} \quad \text{for all } a,b,c \in \mathbb{F}_q.$$

This in turn implies that

$$\psi_k(x^2c - z^2b + 2xza) = \psi_m(c) \quad \text{for all } a,b,c \in \mathbb{F}_q,$$

and so

$$\psi_k(x^2c) = \psi_m(c) \quad \text{for all } c \in \mathbb{F}_q.$$

Thus we must have $kx^2 = m$. Conversely, if $kx^2 = m$ holds, then clearly χ_{ψ_k} and χ_{ψ_m} lie in the same G^F-orbit.

We can now identify two orbits. We will denote the subgroup of squares in $\mathbb{F}_q^\times$ by $\mathbb{F}_q^{\times 2}$. First, assume that $k \in \mathbb{F}_q^{\times 2}$. Then kx^2 is a square, and any square in $\mathbb{F}_q^\times$ can be expressed in this way for some $x \in \mathbb{F}_q^\times$, since $\mathbb{F}_q^{\times 2}$ is a group. Thus $\{\chi_{\psi_k} \mid k \in \mathbb{F}_q^{\times 2}\}$ is an orbit. On the other hand, if k is not a square, then neither is kx^2, and all nonsquares in $\mathbb{F}_q^\times$ can be expressed in this way for some $x \in \mathbb{F}_q^\times$, since $kx^2 = kx'^2 \Rightarrow x^2 = x'^2$, and so we get all $|\mathbb{F}_q^{\times 2}|$ distinct nonsquares. These elementary considerations can be summed up as

$$G^F \backslash \{\chi_{\psi_k} \mid k \in \mathbb{F}_q^\times\} \simeq \mathbb{F}_q^\times / \mathbb{F}_q^{\times 2} \simeq \{\pm 1\}.$$

The above discussion shows that there are $4q$ irreducible representations of dimension $(q^2-1)/2$. Half of them correspond to the orbit $\{\chi_{\psi_k} \mid k \in \mathbb{F}_q^{\times 2}\}$, which

is the one containing χ_{ψ_1}, and the other half correspond to the orbit corresponding to nonsquares in $\mathbb{F}_q^\times$.

3. Some Lemmas

In this section we fix some notation and collect some results from Deligne-Lusztig theory. Most results from the finite field situation ($r=1$) hold for all r. All results in this section except for the last one, were proved for $r=1$ by Deligne and Lusztig (cf. [2], and [4]).

Let $L: G_r \to G_r$, $L(g)=g^{-1}F(g)$ denote the Lang map. We have

$$L^{-1}(xU_r)=\{g\in G_r \mid g^{-1}F(g)\in xU_r\}=X_x.$$

From now on we will denote $H_c^i(X,\overline{\mathbb{Q}}_l)$ simply by $H_c^i(X)$.

Definition. Let G be a finite group that acts on the varieties X and Y. We write $X\sim Y$ if any irreducible representation of G appears in $\sum_{i\geq 0}(-1)^i H_c^i(X)$ if and only if it appears in $\sum_{i\geq 0}(-1)^i H_c^i(Y)$.

Note that the relation just defined is an equivalence relation.

Lemma 3.1. *Suppose that $f: X\to Y$ is a morphism of varieties such that for some $m\geq 0$, the fibre $f^{-1}(y)$ is isomorphic to affine m-space, for all $y\in Y$. Let g,g' be automorphisms of finite order of X,Y such that $fg=g'f$. Then $X\sim Y$.*

Proof. See [4], Lemma 1.9. ∎

As pointed out in [6], for arbitrary r it is enough for representation theoretic purposes to consider varieties X_x where x runs through a set of double coset representatives $U_r\backslash G_r/U_r$. This follows from Lemma 3.1 and the following result.

Lemma 3.2. *The inclusion $L^{-1}(xU_r)\hookrightarrow L^{-1}(U_rxU_r)$ induces an isomorphism*

$$L^{-1}(xU_r)/U_r\cap xU_rx^{-1}\xrightarrow{\sim} L^{-1}(U_rxU_r)/U_r,$$

commuting with the action of G_r^F on both varieties.

Proof. Denote by f the composition of the maps $L^{-1}(xU_r)\hookrightarrow L^{-1}(U_rxU_r)\to L^{-1}(U_rxU_r)/U_r$, where the latter is the natural projection. Clearly f is surjective, because if $gU_r\in L^{-1}(U_rxU_r)/U_r$, with $L(g)\in uxu'$ for $u,u'\in U_r$, then $L(gu)=u^{-1}uxu'F(u)\in xU_r$, so $gu\in L^{-1}(xU_r)$, and $f(gu)=gU_r$.

On the other hand, the fibre of f at gU_r is equal to $\{gv\in L^{-1}(xU_r)\mid v\in U_r\}=\{gv\mid v^{-1}L(g)F(v)\in xU_r,\ v\in U_r\}=\{gv\mid v^{-1}ux\in xU_r, v\in U_r\}=\{gv\mid v^{-1}u\in U_r\cap xU_rx^{-1}\}=\{gv\mid v=u \bmod U_r\cap xU_rx^{-1}\}$. Factoring $L^{-1}(xU_r)$ by $U_r\cap xU_rx^{-1}$ therefore gives an isomorphism, which commutes with the action of G_r^F, by the naturality of f. ∎

Lemma 3.3. *Let $x\in G_r$ be an arbitrary element, and let λ be an element such that $L(\lambda)=x$. Then there is an isomorphism*

$$L^{-1}(xU_r)\xrightarrow{\sim} L^{-1}(F(\lambda)U_rF(\lambda)^{-1}),\quad g\longmapsto g\lambda^{-1},$$

commuting with the action of G_r^F.

Proof. Let $g\in L^{-1}(xU_r)$. Then $L(g\lambda^{-1})=\lambda L(g)F(\lambda)^{-1}\in\lambda xU_rF(\lambda)^{-1}=F(\lambda)x^{-1}xU_rF(\lambda)^{-1}=F(\lambda)U_rF(\lambda)^{-1}$. It is clear that this map is a morphism of varieties, and it has an obvious inverse. ∎

In the case $r = 1$, the Bruhat decomposition says that $B\backslash G/B$ is in bijection with the Weyl group, and that a set of double coset representatives can be taken in $N_G(T)$, the normaliser of T. Suppose that $utwt'u'$ is an arbitrary element in BwB, for $w \in N_G(T)$. Then

$$L^{-1}(Uutwt'u'U) = L^{-1}(Utwt'U) \sim L^{-1}(twt'U) = L^{-1}(t''wU),$$

for some $t'' \in T$. Since $t \mapsto wF(t)w^{-1}$ is a Frobenius map on T, Lang's theorem says that there exists $\lambda \in T$ such that $\lambda^{-1}wF(\lambda)w^{-1} = t''$. The map

$$L^{-1}(t''wU) \longrightarrow L^{-1}(wU), \quad g \longmapsto g\lambda^{-1}$$

is then an isomorphism of varieties, preserving the action of G^F. Indeed, if g is an element in $L^{-1}(t''wU)$, then $L(g\lambda^{-1}) \in \lambda t''wU = wF(\lambda)w^{-1}t''^{-1}t''wUF(\lambda)^{-1} = wF(\lambda)UF(\lambda) = wU$.

Because of this, in the case $r = 1$ it is enough to consider varieties attached to elements of the Weyl group, among all varieties X_x, $x \in G$. This is no longer enough for $r > 1$.

Definition 3.4. Let G be a finite group that acts on the left on the variety X, and let A be a finite abelian group that acts on the right on X. For any character $\theta : A \to \overline{\mathbb{Q}}_l^\times$ and $g \in G$, we define a virtual character of G by

$$\mathscr{L}(g, X)_\theta = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(g, H_c^i(X)_\theta).$$

Here we use the notation $V_\theta = \{v \in V \mid va = \theta(a)v, \text{ for all } a \in A\}$, for V a finite dimensional right $\overline{\mathbb{Q}}_l[A]$-module. Since for any such V we have $V = \bigoplus_\theta V_\theta$, we have a virtual character

$$\mathscr{L}(g, X) = \bigoplus_\theta \mathscr{L}(g, X)_\theta = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(g, H_c^i(X)),$$

classically called the *Lefschetz number*. We will make use of the following results.

Lemma 3.5. *Let G, A, and X be as in the above definition. Then for any $g \in G$ we have*

$$\mathscr{L}(g, X)_\theta = \frac{1}{|A|} \sum_{a \in A} \theta(a^{-1}) \mathscr{L}((g, a), X),$$

where (g, a) acts on X by $x \mapsto gxa$.

Proof. See [1], Proposition 7.2.3. ∎

Lemma 3.6. *Let G and X be as above, and assume that X is a finite set. Then $H_c^i(X) = 0$ if $i \neq 0$, and $H_c^0(X) \simeq \overline{\mathbb{Q}}_l[X]$ is a permutation representation of G. Furthermore the character of this representation is given by $\mathscr{L}(g, X) = |X^g|$.*

Proof. See e.g. [3], Proposition 10.8. ∎

Lemma 3.7. *Let G be a finite group acting on the variety X, and let $X = \coprod_{i \in I} X_i$ be a finite partition of X into disjoint, closed subsets. Assume that G permutes the subsets X_i among them in such a way that the action of G on I is transitive. Let $H = \operatorname{Stab}_G(X_{i_0}) = \{g \in G \mid gX_{i_0} = X_{i_0}\}$ for a fixed $i_0 \in I$. Then the generalised character $g \mapsto \mathscr{L}(g, X)$ of G is induced by the generalised character $h \mapsto \mathscr{L}(h, X_{i_0})$ of H.*

Proof. See [4], Lemma 1.7. ∎

We end with a new result which has no nontrivial analogue for $r = 1$.

Lemma 3.8. *Let $r > 1$, $x \in G_r$, and consider the variety $L^{-1}(xU_r)$. Then the projection map $\rho_r : G_r \to G$ induces an isomorphism*

$$(G_r^1)^F \backslash L^{-1}(xU_r) \xrightarrow{\sim} L^{-1}(\rho_r(x)U),$$

commuting with the action of G_2^F.

Proof. Let f be the map $L^{-1}(xU_r) \to L^{-1}(\rho_r(x)U)$, given by $g \mapsto \rho_r(g)$. This map is surjective because if $g \in L^{-1}(\rho_r(x)U)$, then $i_r(g) \in i_r(L^{-1}(\rho_r(x)U)) \subset L^{-1}(xU_r)$, and $f(i_r(g)) = g$. The fibre of f at g is equal to

$$\{a \cdot i_r(g) \in L^{-1}(xU_r) \mid \rho_r(a) = 1\} = \{a \cdot i_r(g) \mid a \in (G_r^1)^F\},$$

and this shows that factoring $L^{-1}(xU_r)$ by $(G_r^1)^F$ gives an isomorphism which commutes with the action of G_r^F, by the naturality of f. ∎

4. Lusztig's Conjecture for $\mathrm{SL}_2(\mathbb{F}_q[[\varepsilon]]/(\varepsilon^2))$

In the rest of this paper we will focus on the case $G = \mathrm{SL}_2$, K of positive characteristic with q odd, and $r = 2$. In the following we will show that only 4 of the representations of dimension $(q^2-1)/2$ of G_2^F can be realised by varieties of the form X_x for $x \in G_2$.

By Lemma 3.2, all representations that can be realised by varieties X_x, $x \in G_2$, can also be realised by varieties X_x, where x is a representative in $U_2 \backslash G_2 / U_2$. Therefore, the first place to start looking for representations is in the cohomology of varieties corresponding to representatives of $B_2 \backslash G_2 / B_2$. Such a set of representatives is given by

$$\left\{ w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ w_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ e = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \right\}.$$

Proposition 4.1. *Any irreducible representation that can be realised by a variety X_x for $x \in G_2$, can also be realised by a variety X_x, where either $x = w_1, w_2$, or $x = \left(\begin{smallmatrix} 1 & 0 \\ k\varepsilon & 1 \end{smallmatrix}\right)$, for some $k \in \mathbb{F}^\times$.*

Proof. Every element $x \in G_2$ lies in a B_2–B_2 double coset corresponding to one of the elements w_1, w_1, e above. The elements w_1 and w_2 normalise T_2, so as in the case of $r = 1$, for any element $x \in B_2 w_1 B_2 = B_2$ we have $X_x \sim X_{w_1}$, and for any $y \in B_2 w_2 B_2$ we have $X_y \sim X_{w_2}$.

In contrast, the element e does not normalise T_2, so we cannot a priori draw the same conclusions as above. Assume that $x = utet'u'$, where $u, u' \in U_2$ and $t, t' \in T_2$. Then $L^{-1}(utet'u'U_2) \sim L^{-1}(U_2tet'U_2) \sim L^{-1}(tet'U_2)$, and by Lemma 3.3 we have $L^{-1}(tet'U_2) \sim L^{-1}(F(\lambda)U_2F(\lambda)^{-1})$, where $L(\lambda) = tet'$. We can assume that λ has the form

$$\lambda = \begin{pmatrix} t_0 + t_1\varepsilon & 0 \\ u\varepsilon & (t_0+t_1\varepsilon)^{-1} \end{pmatrix} \quad \text{for some } u, t_0 \in \mathbb{F}_q^\times, t_1 \in \mathbb{F}_q.$$

Since we can write $\lambda = e't''$, where $e' = \left(\begin{smallmatrix} 1 & 0 \\ ut_0^{-1}\varepsilon & 1 \end{smallmatrix}\right)$, and $t'' = \left(\begin{smallmatrix} t_0+t_1\varepsilon & 0 \\ 0 & (t_0+t_1\varepsilon)^{-1} \end{smallmatrix}\right)$, we get $L^{-1}(F(\lambda)U_2F(\lambda)^{-1}) = L^{-1}(F(e't'')U_2F(e't'')^{-1}) = L^{-1}(F(e')U_2F(e')^{-1}) \sim L^{-1}(L(e')U_2)$. The element $L(e') = (e')^{-1}F(e')$ is obviously of the form $\left(\begin{smallmatrix} 1 & 0 \\ k\varepsilon & 1 \end{smallmatrix}\right)$, for some $k \in \mathbb{F}^\times$. Thus, for every $x \in B_2eB_2$ we have $X_x \sim L^{-1}(\left(\begin{smallmatrix} 1 & 0 \\ k\varepsilon & 1 \end{smallmatrix}\right) U_2)$, for some $k \in \mathbb{F}^\times$, and any such k appears for some x. ∎

It is clear that the varieties corresponding to elements $\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)$, $k\in\mathbb{F}^\times$, are not all essentially different. Namely, if $\kappa\in G_2$ is such that $L(\kappa)=\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)$, then for any $\left(\begin{smallmatrix}t_0&0\\0&t_0^{-1}\end{smallmatrix}\right)\in T$ we have

$$\begin{aligned}L^{-1}(\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)U_2) &\simeq L^{-1}(F(\kappa)U_2F(\kappa)^{-1}) = L^{-1}\left(F\big[\kappa\left(\begin{smallmatrix}t_0&0\\0&t_0^{-1}\end{smallmatrix}\right)\big]U_2F\big[\kappa\left(\begin{smallmatrix}t_0&0\\0&t_0^{-1}\end{smallmatrix}\right)\big]^{-1}\right)\\ &\simeq L^{-1}\left(L(\kappa\left(\begin{smallmatrix}t_0&0\\0&t_0^{-1}\end{smallmatrix}\right))U_2\right) = L^{-1}\left(\left(\begin{smallmatrix}t_0&0\\0&t_0^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)F\left(\begin{smallmatrix}t_0&0\\0&t_0^{-1}\end{smallmatrix}\right)U_2\right)\\ &= L^{-1}\left(\left(\begin{smallmatrix}t_0&0\\0&t_0^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)\left(\begin{smallmatrix}t_0^q&0\\0&t_0^{-q}\end{smallmatrix}\right)U_2\right) = L^{-1}\left(\begin{pmatrix}t_0^{q-1}&0\\t_0^{q+1}k\varepsilon&t_0^{1-q}\end{pmatrix}U_2\right).\end{aligned}$$

Thus $L^{-1}(\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)U_2)\simeq L^{-1}(\left(\begin{smallmatrix}1&0\\t_0^2k\varepsilon&1\end{smallmatrix}\right)U_2)$, if $t_0^q=t_0$. We see from this that the equivalence class of $L^{-1}(\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)U_2)$ does only depend on the coset of k in $\mathbb{F}^\times/\mathbb{F}_q^{\times 2}$. In the following we will show the stronger result that all varieties X_x for $x=\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)$, $k\in\mathbb{F}^\times$ afford the same irreducible representations.

In [6], 3.3 it is claimed that the variety X_e realises $2q$ irreducible representations of dimension $(q^2-1)/2$. The calculations of Lusztig already show that this cannot be the case, since the variety affords a permutation representation of dimension $q(q^2-1)$ and any permutation representation contains at least one copy of the trivial representation. We will now give the correct decomposition of this permutation representation. Hence we will see that most representations of dimension $(q^2-1)/2$ are not realised in varieties of the form X_x, $x\in G_2$.

For any $k\in\mathbb{F}^\times$ we write $X_k=L^{-1}(\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)U_2)$, by abuse of notation. The variety X_k is endowed with an action of the group $U_2\cap\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)U_2\left(\begin{smallmatrix}1&0\\k\varepsilon&1\end{smallmatrix}\right)^{-1}=U_2^1$, acting by right multiplication. By Lemma 3.1 we have $X_k\sim X_k/U_2^1$.

Theorem 4.2. *Let $k\in\mathbb{F}^\times$. The virtual G_2^F-representation $\sum_{i\geq 0}(-1)^iH_c^i(X_k)$ decomposes into the direct sum of the following representations*

- (*i*) 4 *distinct irreducible representations of dimension $(q^2-1)/2$, each one with multiplicity $(q-1)/2$,*
- (*ii*) *the irreducible representations of dimension* $1,\ q,\ (q+1)/2$*, each with multiplicity one,*
- (*iii*) *the irreducible representations of dimension $q+1$, each with multiplicity* 2.

Moreover, for all $k,k'\in\mathbb{F}^\times$ we have $X_k\sim X_{k'}$.

Proof. The proof goes as follows. First we calculate the variety X_k explicitly following the calculations of Lusztig for the case $k=1$ (cf. [6], 3.3), and show that the representations afforded by X_k can be realised by a finite (0-dimensional) variety $\overline{X}_k$. Next we show that there exists a partition of $\overline{X}_k$ into closed subset such that the action of G_2^F on the parts is transitive. We identify a part $\overline{X}_k^{(\pm1,0)}$ with stabiliser S^F, where S^F is identical to the group $\mathrm{Stab}_{G^F}(\chi_\psi)\cdot N$ in Section 2. Now Lemma 3.7 tells us that the G_2^F-representation afforded by $\overline{X}_k$ is isomorphic to the representation induced from the S^F-representation afforded by the part $\overline{X}_k^{(\pm1,0)}$. Finally, we show how to decompose the latter representation with respect to the action of a certain abelian group, and thanks to the results of Section 2 we can identify exactly which representations of G_2^F occur in the cohomology of X_k.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_2$. The condition that $g \in X_k$ is that

$$F \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k\varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + bk\varepsilon & b + (a + bk\varepsilon)x \\ c + dk\varepsilon & d + (c + dk\varepsilon)x \end{pmatrix},$$

for some $x \in \mathbb{F}[[\varepsilon]]/(\varepsilon^2)$. This condition is equivalent to the system of equations

$$\begin{cases} F(a) = a + bk\varepsilon, \ F(b) = b + (a + bk\varepsilon)x, \\ F(c) = c + dk\varepsilon, \ F(d) = d + (c + dk\varepsilon)x. \end{cases}$$

In order to eliminate x from the equations, we note that since $g \in G_2$, the above system is equivalent to

$$\begin{cases} F(a) = a + bk\varepsilon, \ F(c) = c + dk\varepsilon, \\ (F(b) - b)(c + dk\varepsilon) = (F(d) - d)(a + bk\varepsilon). \end{cases}$$

Setting $a = a_0 + a_1\varepsilon, b = b_0 + b_1\varepsilon, c = c_0 + c_1\varepsilon, d = d_0 + d_1\varepsilon$, we obtain

$$\begin{cases} a_0^q = a_0, \ c_0^q = c_0, \ a_1^q = a_1 + b_0k, \ c_1^q = c_1 + d_0k, \\ (b_0^q - b_0)c_0 = (d_0^q - d_0)a_0, \ (b_1^q - b_1)(c_1 + d_0k) = (d_1^q - d_1)(a_1 + b_0k). \end{cases}$$

Thus, we may identify X_k with the set of all $(a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1) \in \mathbb{F}^8$ such that

(a) $a_0^q = a_0, \ c_0^q = c_0, \ a_1^q = a_1 + b_0k, \ c_1^q = c_1 + d_0k,$
(b) $a_0d_0 - b_0c_0 = 1, \ a_0d_1 + a_1d_0 - b_0c_1 - b_1c_0 = 0 \ (\Leftrightarrow \det g = 1),$
(c) $(b_0^q - b_0)c_0 = (d_0^q - d_0)a_0,$
$b_1^qc_0 - b_1c_0 + b_0^qc_1 - b_0c_1 + b_0^qd_0k = d_1^qa_0 + d_1^qa_0 - a_0d_1 + d_0^qa_1 - d_0a_1 + d_0^qb_0k.$

Now the first equation (c) follows by raising the first equation (b) to the qth power, and using the two first equations (a). The second equation (c) follows by raising the second equation (b) to the qth power, using the two last equations (a), and adding again the second equation (b). Thus the equations (c) can be omitted.

The first equation (b) can be written (using (a)):

$$a_0(c_1^q - c_1)k^{-1} - c_0(a_1^q - a_1)k^{-1} = 1, \text{ that is } (a_0c_1 - c_0a_1)^q - (a_0c_1 - c_0a_1) = k.$$

Setting $f = a_0c_1 - c_0a_1$, we see that X_k can be identified with the set of all $(a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, f) \in \mathbb{F}^9$ such that

$$\begin{cases} a_0^q = a_0, \ c_0^q = c_0, \ a_1^q = a_1 + b_0k, \ c_1^q = c_1 + d_0k, \\ f^q - f = k, \ f = a_0c_1 - c_0a_1, \ a_0d_1 + a_1d_0 - b_0c_1 - b_1c_0 = 0. \end{cases}$$

We now factor out by the action of U_2^1. If $u = \left(\begin{smallmatrix} 1 & x\varepsilon \\ 0 & 1 \end{smallmatrix}\right) \in U_2^1$, then the action $g \mapsto gu$ on X_k is given in terms of coordinates by

$$(a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, f) \longmapsto (a_0, b_0, c_0, d_0, a_1, b_1 + a_0x, c_1, d_1 + c_0x, f).$$

Suppose that $(a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, f)$ and $(a_0, b_0, c_0, d_0, a_1, b_1', c_1, d_1', f)$ are two points on X_k. Then

$$a_0d_1 + a_1d_0 - b_0c_1 - b_1c_0 = 0, \text{ and } a_0d_1' + a_1d_0 - b_0c_1 - b_1'c_0 = 0.$$

These equations imply that $a_0(d_1 - d_1') = c_0(b_1 - b_1')$, which is equivalent to $b_1' = b_1 + a_0x$, $d_1' = d_1' + c_0x$, for some $x \in \mathbb{F}$. Thus the quotient variety X_k/U_2^1 may be identified with the set of points $(a_0, b_0, c_0, d_0, a_1, c_1, f) \in \mathbb{F}^7$ such that

$$a_0^q = a_0, \ c_0^q = c_0, \ a_1^q = a_1 + b_0k, \ c_1^q = c_1 + d_0k, \ f^q - f = k, \ f = a_0c_1 - c_0a_1.$$

This in turn is naturally isomorphic to

$$\{(a_0, c_0, a_1, c_1, f) \in \mathbb{F}^5 \mid a_0^q = a_0,\ c_0^q = c_0,\ f^q - f = k,\ f = a_0c_1 - c_0a_1\}.$$

We consider the obvious projection $\alpha : (a_0, c_0, a_1, c_1, f) \mapsto (a_0, c_0, f)$ of this set, to the finite set

$$\overline{X}_k = \{(a_0, c_0, f) \in \mathbb{F}^3 \mid a_0^q = a_0,\ c_0^q = c_0,\ f^q - f = k,\ (a_0, c_0) \neq (0, 0)\}.$$

We remark that if $f^q - f = k$ for some $f \in \mathbb{F}$, then $(f + f_0)^q - (f + f_0) = k$ for any $f_0 \in \mathbb{F}_q$. Hence, for any $k \in \mathbb{F}^\times$, the equation $f^q - f = k$ has q solutions, and if f is such a solution, then every solution is of the form $f + f_0$, for some $f_0 \in \mathbb{F}_q$.

Define an action of G_2^F on $\overline{X}_k$ as the unique action such that $\alpha(gx) = g\alpha(x)$, for $x \in X_k$, $g \in G_2^F$. The fibre of α at $(a_0, c_0, f) \in \overline{X}_k$ is the affine line $\{(a_1, c_1) \in \mathbb{F}^2 \mid a_0c_1 - c_0a_1 = f\}$. Thus by Lemma 3.1, $X_k \sim X_k/U_2^1 \sim \overline{X}_k$. Since $\overline{X}_k$ is a 0-dimensional variety, it follows from Lemma 3.6 that $\sum_{i\geq 0}(-1)^i H_c^i(\overline{X}_k) = H_c^0(\overline{X}_k) \simeq \overline{\mathbb{Q}}_l[\overline{X}_k]$, which is a permutation representation of dimension $|\overline{X}_k| = q(q^2 - 1)$.

We now turn to the problem of decomposing the representation $\overline{\mathbb{Q}}_l[\overline{X}_k]$ into irreducibles. Consider the group

$$A = \left\{\begin{pmatrix} \pm 1 & 0 \\ x\varepsilon & \pm 1 \end{pmatrix} \mid x \in \mathbb{F}_q\right\}.$$

This group acts on X_k by right multiplication. There is a unique action of A on $\overline{X}_k$ satisfying $\alpha(xa) = \alpha(x)a$, for $x \in X_k$, $a \in A$. The action on $\overline{X}_k$ is given in terms of coordinates by

$$(a_0, c_0, f) \longmapsto (\pm a_0, \pm b_0, f + x).$$

The set of orbits $\overline{X}_k/A$ defines a partition of $\overline{X}_k$ into closed subsets $\overline{X}_k^{(a_0,c_0)}$, indexed by pairs $(a_0, c_0) \in \mathbb{F}_q^2/\{\pm 1\}$, $(a_0, c_0) \neq (0, 0)$. Hence, each orbit contains $2q$ elements.

Now consider the action of the group G_2^F on $\overline{X}_k$. For $g = \left(\begin{smallmatrix} x_0+x_1\varepsilon & y_0+y_1\varepsilon \\ z_0+z_1\varepsilon & w_0+w_1\varepsilon \end{smallmatrix}\right) \in G_2^F$, the action is given in terms of coordinates by

$$\begin{aligned}&(a_0, c_0, f) \longmapsto (x_0a_0 + y_0c_0, z_0a_0 + w_0c_0,\\ &f + a_0^2(x_0z_1 - z_0x_1) + a_0c_0(x_0w_1 + y_0z_1 - z_0y_1 - w_0x_1) + c_0^2(y_0w_1 - w_0y_1)).\end{aligned}$$

Thus $g\overline{X}_k^{(a_0,c_0)} = \overline{X}_k^{(x_0a_0+y_0c_0, z_0a_0+w_0c_0)}$, and G_2^F acts transitively on the set of orbits $\overline{X}_k^{(a_0,c_0)}$. The stabiliser of the orbit $\overline{X}_k^{(\pm 1,0)}$ is given by

$$S^F := \mathrm{Stab}_{G_2^F}(\overline{X}_k^{(\pm 1,0)}) = \left\{\begin{pmatrix} \pm 1 + x_1\varepsilon & y_0 + y_1\varepsilon \\ z_1\varepsilon & \pm 1 + w_1\varepsilon \end{pmatrix} \in G_2^F\right\}.$$

It follows from Lemma 3.7 that the G_2^F-representation $\sum_{i\geq 0}(-1)^i H_c^i(X_k) \simeq H_c^0(\overline{X}_k)$ is induced by the S^F-representation $\sum_{i\geq 0}(-1)^i H_c^i(\overline{X}_k^{(\pm 1,0)}) = H_c^0(\overline{X}_k^{(\pm 1,0)})$. We will determine the latter by using some character theory.

Let s denote an element in S^F, and let $\chi : A \to \overline{\mathbb{Q}}_l^\times$ be a character. Since induction preserves direct sums, we have

$$H_c^0(\overline{X}_k) = \bigoplus_{\chi} \mathrm{Ind}_{S^F}^{G_2^F} H_c^0(\overline{X}_k^{(\pm 1,0)})_\chi,$$

and by Lemma 3.5 and Lemma 3.6 we have

$$\mathrm{Tr}(s, H_c^0(\overline{X}_k^{(\pm 1,0)})_\chi) = \frac{1}{|A|}\sum_{a\in A}\chi(a^{-1})|(\overline{X}_k^{(\pm 1,0)})^{(s,a)}|.$$

Thus we have to determine the fixed points of $\overline{X}_k^{(\pm 1,0)}$ under the action of $(s,a)\in S^F\times A$. Every $s\in S^F$ can be decomposed uniquely as $s=ua'$, where $u\in\{\left(\begin{smallmatrix}1+x\varepsilon & y_0+y_1\varepsilon\\ 0 & 1-x\varepsilon\end{smallmatrix}\right)\in G_2^F\}$, and $a'\in A$. The set $\overline{X}_k^{(\pm 1,0)}$ is fixed under conjugation by elements in A, and the element u leaves it fixed. Thus

$$|(\overline{X}_k^{(\pm 1,0)})^{(s,a)}| = \begin{cases} 2q & \text{if } a' = a^{-1},\\ 0 & \text{otherwise,}\end{cases}$$

and so

$$\mathrm{Tr}(s, H_c^0(\overline{X}_k^{(\pm 1,0)})_\chi) = \frac{1}{2q}\chi(a')2q = \chi(a').$$

We see that the character of the representation $H_c^0(\overline{X}_k^{(\pm 1,0)})$ is the direct sum of all characters of S^F given by

$$\begin{pmatrix}\pm 1 + x_1\varepsilon & y_0+y_1\varepsilon\\ z_1\varepsilon & \pm 1 + w_1\varepsilon\end{pmatrix}\longmapsto \chi^{\pm}(\pm 1)\cdot\psi(z_1),$$

where $\chi^{\pm}\in\mathrm{Hom}(\{\pm 1\},\overline{\mathbb{Q}}_l^\times)$, $\psi\in\mathrm{Hom}(\mathbb{F}_q^+,\overline{\mathbb{Q}}_l^\times)$. Comparing this with the description of the irreducible representations of G_2^F of dimension $(q^2-1)/2$ given in Section 2, we see that there are four characters χ for which the induced characters of G_2^F are distinct and irreducible. All other characters χ such that $\chi^2\neq 1$ give rise to representations of G_2^F isomorphic to one of these. Next consider $\chi = 1$. Inducing the trivial character of S^F to G_2^F gives the character of the permutation representation $\overline{\mathbb{Q}}_l[G_2^F/S^F]\simeq\overline{\mathbb{Q}}_l[G^F/\{\left(\begin{smallmatrix}\pm 1 & y_0\\ 0 & \pm 1\end{smallmatrix}\right)\}]$, and from the classical finite field case we know that this representation consists of the representations of dimension 1 and q, and $(q-3)/2$ irreducible representations of dimension $q+1$. Analogously, inducing the character χ for which $\chi^2=1,\ \chi\neq 1$, we see that the resulting representation consists of the two irreducible representations of dimension $(q+1)/2$, and $(q-3)/2$ irreducible representations of dimension $q+1$ isomorphic to the ones corresponding to $\chi = 1$.

This discussion shows that the above results are independent of the choice of $k\in\mathbb{F}^\times$. Hence, the theorem is proved. ∎

5. Realising The Missing Representations

Using the observations of the previous sections, we show how all irreducible representations of dimension $(q^2-1)/2$ of G_2^F can be realised in the cohomology of a certain variety.

Consider the subgroup $S = ZG_2^1U$ of G_2, where Z is the centre of G_2. Then it is clearly an F-stable subgroup, and S^F is compatible with the notation used in the previous section. As we have seen, all irreducible representations of dimension $(q^2-1)/2$ of G_2^F can be obtained by inducing certain 1-dimensional representations of the subgroup S^F. Some elementary calculations show that the commutator subgroup of S is given by

$$(S,S) = \left\{\begin{pmatrix}1+x\varepsilon & y\varepsilon\\ 0 & 1-x\varepsilon\end{pmatrix}\right\},$$

and this is again an F-stable subgroup. Consider the variety

$$Y = \{g \in G_2 \mid g^{-1}F(g) \in (S,S)\}/(S,S),$$

where (S,S) acts by right multiplication. The variety Y is acted on by G_2^F and S^F by left and right multiplication, respectively.

Since (S,S) is F-stable, there is an obvious inclusion map $Y \to G_2/(S,S)$, whose image is $(G_2/(S,S))^F \simeq G_2^F/(S,S)^F$. We thus have an isomorphism $Y \simeq G_2^F/(S,S)^F$.

Let χ be a character of S^F such that χ is an extension of a character $\left(\begin{smallmatrix} 1 & 0 \\ x\varepsilon & 1 \end{smallmatrix}\right) \mapsto \psi(x)$, for some $\psi \in \mathrm{Hom}(\mathbb{F}_q^+, \overline{\mathbb{Q}}_l^\times)$. We have seen that there are two distinct orbits of such characters χ, with respect to the action of G^F. Let θ be an arbitrary character of the subgroup $\{\left(\begin{smallmatrix} \pm 1 & y_0 \\ 0 & \pm 1 \end{smallmatrix}\right) \in G^F\}$.

Proposition 5.1. *Every irreducible representation of G_2^F of dimension $(q^2-1)/2$ is given by*

$$\sum_{i \geq 0} (-1)^i (H_c^i(Y) \otimes \chi \otimes \theta) = H_c^0(Y) \otimes \chi \otimes \theta,$$

where the characters χ and θ are identified with their corresponding 1-dimensional representations of S^F. Moreover, if χ runs through a set of representatives of orbits under G^F, then each irreducible representation of dimension $(q^2-1)/2$ appears exactly once.

Proof. We have seen above that $Y \simeq G_2^F/(S,S)^F$, so by Lemma 3.6 the cohomology of Y is concentrated in degree 0, and $H_c^0(Y) \simeq \overline{\mathbb{Q}}_l[G_2^F/(S,S)^F]$. Now let V be a representation of S^F that factors through $(S,S)^F$, i.e., a 1-dimensional representation. Then there is a representation of G_2^F given by $H_c^0(Y) \otimes_{\overline{\mathbb{Q}}_l[S^F]} V$, and this representation is isomorphic to the one given by inducing V to G_2^F (cf. [3], ch. 4). By the description of the irreducible representations of dimension $(q^2-1)/2$ as induced by 1-dimensional representations of S^F given in Section 2, it follows that these representations are given by $H_c^0(Y) \otimes \chi \otimes \theta$, as asserted. ∎

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